A new Hamiltonian for a massive relativistic particle with spin one in a generalized Heisenberg/Schrödinger picture

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Abstract

We consider a particular 4-dimensional generalization of the transition from the Heisenberg to the Schrödinger picture. The space-time independent expansion with respect to the unitary irreducible representations of the Lorentz group is applied, as Fourier transformation in the Heisenberg picture, to the states of a massive relativistic particle. A new Hamiltonian operator has been found for such a particle with spin one.

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I. INTRODUCTION

In this paper we present a new mathematical formalism for describing a massive relativistic particle with spin one. In this formalism, we use a four-dimensional transition from the Heisenberg to the Schrödinger picture. In quantum mechanics, the transition from the Heisenberg to the Schrödinger picture is carried out by the unitary transformation $S(t) = \exp(-itH)$, where H is the Hamiltonian operator of the particle; (we choose here a system of units such that $\hbar = 1$, $c = 1$). The state of a particle in the Heisenberg picture and the particle operators in the Schrödinger picture are defined as time-independent functions and operators, respectively. In our earlier work [1], we generalized this transition to the transformation

$$S(t, \mathbf{x}) = \exp[-i(tH - \mathbf{x} \cdot \mathbf{P})], \tag{1.1}$$

where $\mathbf{P}$ is the momentum operator of the particle. In this context, the functions in the Heisenberg picture and the operators in the Schrödinger picture are independent of the time and space coordinates t, $\mathbf{x}$. The Fourier transform of the state in the Heisenberg picture must be independent of the space-time coordinates. That is why the plane waves $\sim \exp(i\mathbf{x} \cdot \mathbf{p})$ cannot be applied in this Fourier transformation. Accordingly, the momentum and the Hamiltonian operator of the particle cannot be expressed in terms of the spatial derivative $-i\nabla_{\mathbf{x}}$. Under these premises, there is no $\mathbf{x}$-representation. As a result, the plane waves in the new Schrödinger picture and also the space-time coordinates in the operators of the new Heisenberg picture appear in different representations. In the Heisenberg picture at first one can use the momentum representation and subsequently the representation which is defined via a space-time independent Fourier transformation.

Let the function $\Psi^{(s)}(\mathbf{p})$ be a relativistic wave function of a particle in the momentum representation ($\mathbf{p}$ = momentum, m = mass, $p_0 := \sqrt{m^2 + \mathbf{p}^2}$, s = spin). In the context of the generalization $S(t) \Rightarrow S(t, \mathbf{x})$, the function $\Psi^{(s)}_{\sigma}(\mathbf{p})$ is a wave function in the Heisenberg picture. Under the Lorentz transformation g with boost and rotation generators [2–5]

$$\mathbf{N}(\mathbf{p},\mathbf{s}) := ip_0\nabla_{\mathbf{p}} - \frac{\mathbf{s}\times\mathbf{p}}{p_0+m}, \quad \mathbf{J}(\mathbf{p},\mathbf{s}) := -i\mathbf{p}\times\nabla_{\mathbf{p}} + \mathbf{s} := \mathbf{L}(\mathbf{p}) + \mathbf{s}, \tag{1.2}$$

and parameters $\mathbf{u}, u_0$, with $(\mathbf{u}^2 - u_0^2 = 1)$, the function $\Psi_\mu^{(s)}(\mathbf{p})$ transforms by the unitary representation (μ= spin projection)

$$T_g \Psi_\mu^{(s)}(\mathbf{p}) = \sum_{\mu'=-s}^{s} W_{\mu\mu'}^{(s)}(\mathbf{p},\mathbf{u})\, \Psi_{\mu'}^{(s)}(g^{-1}\mathbf{p}), \tag{1.3}$$

where $W_{\mu\mu'}^{(s)}(\mathbf{p},\mathbf{u})$ are the Wigner functions (σ are the Pauli matrices)

$$W^{(1/2)}(\mathbf{p},\mathbf{u}) = \frac{(p_0+m)(u_0+1) - \mathbf{u}\cdot\mathbf{p} + i\sigma(\mathbf{p}\times\mathbf{u})}{\sqrt{2(u_0+1)(p_0+m)(p_0u_0 - \mathbf{p}\cdot\mathbf{u} + m)}}, \quad W^{(1/2)}(\mathbf{p},\mathbf{u})\, W^{\dagger(1/2)}(\mathbf{p},\mathbf{u}) = 1. \tag{1.4}$$

Such a wave function has positive definite norm

$$\int \frac{d\mathbf{p}}{p_0} \sum_{\mu=-s}^{s} |\Psi_\mu^{(s)}(\mathbf{p})| = \int \frac{d\mathbf{p}}{p_0} \sum_{\mu=-s}^{s} |\Psi_\mu^{(s)}(\mathbf{p},t,\mathbf{x})| < \infty, \tag{1.5}$$

and can be expanded with respect to irreducible unitary representations of the Lorentz group. This function is covariant only with respect to the set of spin and momentum variables, and not with respect to each of them separately. In (1.5), the function $\Psi^{(s)}(\mathbf{p},t,\mathbf{x}) := S(t,\mathbf{x})\,\Psi^{(s)}(\mathbf{p})$ is the wave function in the new Schrödinger picture in momentum representation. The relativistic spinors, transforming by non unitary finite representations of the Lorentz group, have not definite norms. In this case these spinors are not useful.

The unitary representations correspond to the eigenvalues $1+\alpha^2-\lambda^2$ of the first $C_1(\mathbf{p}) := \mathbf{N}^2 - \mathbf{J}^2$ and the eigenvalues $\alpha\lambda$ of the second Casimir operator $C_2(\mathbf{p}) = \mathbf{N}\cdot\mathbf{J}$, $(0 \le \alpha < \infty, \quad \lambda = -s, ..., s)$. The range of α defines the fundamental series of the unitary representations. The formalism of harmonic analysis on the Lorentz group has been used by many authors (a detailed list of references can be found in Ref. [9–12] e.g.). The four-dimensional generalization of the Heisenberg/Schrödinger picture introduces new features into the nature of the description of particle states. It is necessary to develop a mathematical formalism in the framework of this approach for describing the relativistic particles. In [1,13]

the space-time independent expansion with respect to the unitary irreducible representation of the Lorentz group was applied as Fourier transformation in the Heisenberg picture for the relativistic particle with spin 0 and spin 1/2. This procedure will be applied here for such a practically important example as the massive particle with spin 1. We shall show that the consistent determination of the wave functions of a particle in the Schrödinger picture requires the use of the wave functions in the momentum representation or of the matrix elements of the fundamental series of unitary representations of the Lorentz group. Since the operators $-i\nabla_{\mathbf{x}}$ are not momentum operators, we want to find the Hamiltonian and the momentum operators for the particle with spin 1 . In this paper, the operators we obtain are expressed through the group parameter α. At first we will give a short review for the description of particles with spin 0 in the context of the application of (1.1).

II. SPIN 0 - PARTICLE

In this case, the operator $C_1(\mathbf{p})$ have the eigenfunctions

$$\xi^{(0)}(\mathbf{p},\alpha,\mathbf{n}) := [(p_0 n_0 - \mathbf{p}\cdot\mathbf{n})/m]^{-1+i\alpha}, \tag{2.1}$$

where $(n_0,\,\mathbf{n})$ is a null vector $n_0^2 - \mathbf{n}^2 = 0$.

The Fourier transforms for the states of the relativistic particle with spin 0 in terms the basis functions (2.1) ($\mathbf{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, $\quad d\omega_{\mathbf{n}} = \sin\theta d\theta d\varphi$, have the form [14],

$$\Psi^{(0)}(\mathbf{p}) = \frac{1}{(2\pi)^{3/2}} \int \alpha^2 d\alpha\, d\omega_{\mathbf{n}}\, \Psi^{(0)}(\alpha,\mathbf{n})\, \xi^{(0)}(\mathbf{p},\alpha,\mathbf{n}), \tag{2.2}$$

$$\Psi^{(0)}(\alpha,\mathbf{n}) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{p_0}\, \Psi^{(0)}(\mathbf{p})\, \xi^{*(0)}(\mathbf{p},\alpha,\mathbf{n}). \tag{2.3}$$

The functions $\Psi^{(0)}(\mathbf{p})$ and $\Psi^{(0)}(\alpha,\mathbf{n})$ are the state functions of the particle in $\mathbf{p}$- and in the $(\alpha,\mathbf{n})$-representation. The completeness and orthogonality relations for the functions $\xi^{(0)}(\mathbf{p},\alpha,\mathbf{n})$, $\quad \xi^{*(0)}(\mathbf{p},\alpha,\mathbf{n})$ are given in the Appendix. In the $(\alpha,\mathbf{n})$-representation the free

Hamiltonian and the momentum operators for a particle with spin 0 are the differential-difference operators [10] ($\mathbf{L} := \mathbf{L}(\theta, \varphi)$)

$$H^{(0)}(\alpha, \mathbf{n}) = m\left[\cosh(i\frac{\partial}{\partial\alpha}) + \frac{i}{\alpha}\sinh(i\frac{\partial}{\partial\alpha}) + \frac{\mathbf{L}^2}{2\alpha^2}\exp(i\frac{\partial}{\partial\alpha})\right], \tag{2.4}$$

$$\mathbf{P}^{(0)}(\alpha, \mathbf{n}) = \mathbf{n}\left[H^{(0)}(\alpha, \mathbf{n}) - m\exp(i\frac{\partial}{\partial\alpha})\right] - m\frac{\mathbf{n}\times\mathbf{L}}{\alpha}\exp(i\frac{\partial}{\partial\alpha}). \tag{2.5}$$

The eigenfunction of this operators is $\xi^{(0)}_{\mathbf{p}}(\alpha, \mathbf{n}) := \xi^{*(0)}(\mathbf{p}, \alpha, \mathbf{n})$:

$$H^{(0)}(\alpha, \mathbf{n})\,\xi^{(0)}_{\mathbf{p}}(\alpha, \mathbf{n}) = p_0\,\xi^{(0)}_{\mathbf{p}}(\alpha, \mathbf{n}), \quad \mathbf{P}^{(0)}(\alpha, \mathbf{n})\,\xi^{(0)}_{\mathbf{p}}(\alpha, \mathbf{n}) = \mathbf{p}\,\xi^{(0)}_{\mathbf{p}}(\alpha, \mathbf{n}). \tag{2.6}$$

The operators in (2.4)-(2.6) are used for the relastivistic description of the two-body problem [10,15–18]. In this case the vector $\mathbf{q} = \alpha\mathbf{n}/m$ is applied. In the nonrelativistic limit

$$C^{(0)}_1(\mathbf{p}) \to -m^2\nabla^2_{\mathbf{p}}, \quad \xi^{(0)}(\mathbf{p}, \alpha, \mathbf{n}) \;\to\; \exp(-i\alpha\mathbf{n}\cdot\mathbf{p}/m), \tag{2.7}$$

$$H^{(0)}(q, \mathbf{n}) - m \quad\to\quad -\frac{1}{2mq^2}\frac{\partial}{\partial q}q^2\frac{\partial}{\partial q} + \frac{\mathbf{L^2}}{2mq^2}, \quad \mathbf{P}^{(0)}(q, \mathbf{n}) \to \quad -i\nabla_{\mathbf{q}}. \tag{2.8}$$

The functions $\exp(i\alpha\mathbf{n}\cdot\mathbf{p}/m)$ realize the unitary irreducible representations of the Galileo group:

$$\Psi(\alpha\mathbf{n}) = \frac{1}{(2\pi)^{3/2}}\int d\mathbf{p}\Psi(\mathbf{p})\exp(i\alpha\mathbf{n}\cdot\mathbf{p}/m). \tag{2.9}$$

Since Wigner, particle are associated with unitary representation of the Poincaré group. If one introduces the generators of the Lorentz algebra $\mathbf{N}(\alpha, \mathbf{n})$ for the particle with spin 0

$$\mathbf{N}^{(0)}(\alpha, \mathbf{n}) := \alpha\mathbf{n} + (\mathbf{n}\times\mathbf{L} + \mathbf{L}\times\mathbf{n})/2, \tag{2.10}$$

then [1], instead of the vector $\alpha\mathbf{n}$, the $(\alpha, \mathbf{n})$-representation can be recognized as a representation of the Poincaré group. The operators $H^{(0)}(\alpha, \mathbf{n})$, $\mathbf{P}^{(0)}(\alpha, \mathbf{n})$, $\mathbf{N}^{(0)}(\alpha, \mathbf{n})$, $\mathbf{L}$ satisfy the commutation relations of the Poincaré algebra.

The Casimir operator $C_1(\mathbf{p})$, and the functions $\xi^{(0)}(\mathbf{p}, \alpha, \mathbf{n})$ do not depend on the space-time coordinates $\mathbf{x}$, t. That is why the functions in the expansions (2.2), (2.3) and the

operators in (2.4)-(2.6) are independent on the space-time coordinates likewise. In the framework of the four-dimensional generalization of the Heisenberg to the Schrödinger picture $S(t)\Rightarrow S(t,\mathbf{x})$, the functions in (2.2), (2.3) and the operators in the (2.4)-(2.6) must be seen as functions in the Heisenberg picture and, accordingly, as operators in the Schrödinger picture for the particles with spin 0. If the transformation (1.1) is not applied, then we have no possibility to introduce the plane wave $\exp(i\mathbf{x}\cdot\mathbf{p})$ into the relativistic state functions (2.2), (2.3). In the nonrelativistic limit there is such a possibility. The function $\exp(i\alpha\mathbf{n}\cdot\mathbf{p}/m)$ in (2.9) has the form of the plane wave $\exp(i\mathbf{x}\cdot\mathbf{p})$. Thus, if the generalization (1.1) is not applied, the function $\exp(i\alpha\mathbf{n}\cdot\mathbf{p}/m)$ can be replaced by $\exp(i\mathbf{x}\cdot\mathbf{p})$. In such a form, the plane waves can be introduced in the Schrödinger picture through the Fourier transformation, and then an $\mathbf{x}$-representation can be introduced. In the relativistic expansion (2.3) this method cannot be used. The functions $\exp(i\mathbf{x}\cdot\mathbf{p})$, $\xi^{*(0)}(\mathbf{p},\alpha,\mathbf{n})$ have different forms.

The application of the transformation (1.1) gives the state of the particle in the Schrödinger picture in bfp and $(\alpha,\mathbf{n})$ -representation:

$\Psi^{(0)}(\mathbf{p},t,\mathbf{x})=S(t,\mathbf{x})\Psi^{(0)}(\mathbf{p}),\quad \Psi^{(0)}(\alpha,\mathbf{n},t,\mathbf{x})=S(t,\mathbf{x})\Psi^{(0)}(\alpha,\mathbf{n}).$

The Fourier expansion $(\exp[-i(p{\cdot}x)]:=\exp[-i(tp_0-\mathbf{x}\cdot\mathbf{p})])$

$$\Psi^{(0)}(\alpha,\mathbf{n},t,\mathbf{x})=\frac{1}{(2\pi)^{3/2}}\int\frac{d\mathbf{p}}{p_0}\,\Psi^{(0)}(\mathbf{p})\,\xi^{(0)}_{\mathbf{p}}(\alpha,\mathbf{n})\,\exp[-i(p\cdot x)], \tag{2.11}$$

in contrast to the usual expansion

$$\Psi^{(0)}(t,\mathbf{x})=\frac{1}{(2\pi)^{3/2}}\int\frac{d\mathbf{p}}{p_0}\Psi^{(0)}(\mathbf{p})\,\exp[-i(p\cdot x)], \tag{2.12}$$

contains the matrix elements $\xi^{(0)}_{\mathbf{p}}(\alpha,\mathbf{n})$ of the unitary representation of the Lorentz group.

In the expansion (2.12), the plane waves in the form $\sim$ constexp $-i(p{\cdot}x)$ appear as the wave functions of the particle with the definite momentum $\mathbf{p}$ and spin 0. Similar expansion for the particle with spin 1/2 or spin 1 contain the Dirac bispinor (spin 1/2) or the unit polarization 4- vector (spin 1), respectively. An important difference between (2.11) and (2.12) is that the plane waves $\sim$ constexp$[-i(p{\cdot}x)]$ without the wave functions in the Heisenberg picture in accordance with the transformation (1.1) cannot express the wave functions of the particle.

The wave functions with definite momentum in the Schrödinger picture in $(\alpha, \mathbf{n})$-representation are the functions

$$\xi^{(0)}_{\mathbf{p}}(\alpha, \mathbf{n}, t, \mathbf{x}) = \xi^{(0)}_{\mathbf{p}}(\alpha, \mathbf{n}) \exp[-i(p{\cdot}x)], \tag{2.13}$$

in (2.11).

The expression (2.12) and the similar expansions for the particle with spin 1/2 or spin 1 containing the Dirac bispinor or the unit polarization 4- vector are not transformations from one representation to another.

In the nonrelativistic limit for the Fourier expansion in the Schrödinger picture we have

$$\Psi(\alpha\mathbf{n}, t, \mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int d\mathbf{p}\Psi(\mathbf{p}) \exp(i\alpha\mathbf{n}\cdot\mathbf{p}/m) \exp[-i(tp^2/2m - \mathbf{x}\cdot\mathbf{p})]. \tag{2.14}$$

III. SPIN 1 - PARTICLE

The expansions (2.2), (2.3) are generalized in [19,20] for the particle with spin. They can be expressed in the form

$$\Psi^{(s)}_{\mu}(\mathbf{p}) = \frac{1}{(2\pi)^{3/2}} \sum_{\mu'=-s}^{s} \int ({\mu'}^2 + \alpha^2) d\alpha\, d\omega_{\mathbf{n}}\, D^{(s)}_{\mu\mu'}(R_w)\, \xi^{(0)}(\mathbf{p}, \alpha, \mathbf{n})\, \Psi^{(s)}_{\mu'}(\alpha, \mathbf{n}), \tag{3.1}$$

$$\Psi^{(s)}_{\mu}(\alpha, \mathbf{n}) = \frac{1}{(2\pi)^{3/2}} \sum_{\mu'=-s}^{s} \int \frac{d\mathbf{p}}{p_0}\, D^{\dagger(s)}_{\mu\mu'}(R_w)\, \xi^{*(0)}(\mathbf{p}, \alpha, \mathbf{n})\, \Psi^{(s)}_{\mu'}(\mathbf{p}), \tag{3.2}$$

where $\Psi^{(s)}_{\mu}(\alpha, \mathbf{n})$ is the wave function in $(\alpha, \mathbf{n})$ represantion and the matrix $D^{(s)}(R_w)$ must have the qualities of the Wigner rotation in (1.3), (1.4).

In [13] this matrix (spin=1/2) has been found by means of the solutions of the eigenvalue equations of the operator $C_1(\mathbf{p})$

$$D^{(1/2)}(R_w) := D^{(1/2)}(\mathbf{p}, \mathbf{n}) = \frac{p_0 - \mathbf{p}\cdot\mathbf{n} + m - i\sigma\cdot(\mathbf{p}{\times}\mathbf{n})}{\sqrt{2(p_0+m)(p_0 - \mathbf{p}\cdot\mathbf{n})}}, \quad D^{(1/2)}(\mathbf{p}, \mathbf{n})\, D^{\dagger(1/2)}(\mathbf{p}, \mathbf{n}) = 1. \tag{3.3}$$

In the $(\alpha, \mathbf{n})$-representation the functions:

$$\xi^{(1/2)}_{\mathbf{p}}(\alpha,\mathbf{n}) := D^{\dagger(1/2)}(\mathbf{p},\mathbf{n})\,\xi^{(0)}_{\mathbf{p}}(\alpha,\mathbf{n}), \tag{3.4}$$

were determined as the eigenfunctions of the Hamiltonian and the momentum operator for the particle with spin 1/2. In this case

$$\mathbf{J} := \mathbf{L} + \mathbf{s}, \quad \mathbf{N} := \alpha\mathbf{n} + (\mathbf{n}\times\mathbf{J} + \mathbf{J}\times\mathbf{n})/2. \tag{3.5}$$

and $C_1(\alpha,\mathbf{n}) = 1 + \alpha^2 - (\mathbf{s}\cdot\mathbf{n})^2, \quad C_2(\alpha,\mathbf{n}) = \alpha\mathbf{s}\cdot\mathbf{n}$.

For the particle with spin one, we use the eigenfunctions $\xi^{(1)}(\mathbf{p},\alpha,\mathbf{n})$ of both Casimir operators $C_1(\mathbf{p})$ and $C_2(\mathbf{p})$:

$$\xi^{(1)}(\mathbf{p},\alpha,\mathbf{n}) = D^{(1)}(\mathbf{p},\mathbf{n})\,D(\mathbf{n})\,\xi^{(0)}(\mathbf{p},\alpha,\mathbf{n}). \tag{3.6}$$

The matrix $D^{(1)}(\mathbf{p},\mathbf{n})$ can be obtained from the matrix (3.3) and the Clebsch-Gordan coefficients. The matrix $D^{\dagger}(\mathbf{n})$, $(D(\mathbf{n})D^{\dagger}(\mathbf{n}) = 1)$ contains the eigenfunctions of the operator $\mathbf{s}\cdot\mathbf{n}$, with the eigenvalues λ=-1, 0, 1.

In order to define the Hamiltonian and the momentum operators, we consider the functions

$$\xi^{(1)}_{\mathbf{p}}(\alpha,\mathbf{n}) := D^{\dagger}(\mathbf{n})\,D^{\dagger(1)}(\mathbf{p},\mathbf{n})\,\xi^{(0)}_{\mathbf{p}}(\alpha,\mathbf{n}), \tag{3.7}$$

as states of the free particle with spin 1 with a definite momentum in the Heisenberg picture in $(\alpha,\mathbf{n})$-representation

$$H^{(1)}(\alpha,\mathbf{n})\,\xi^{(1)}_{\mathbf{p}}(\alpha,\mathbf{n}) = p_0\,\xi^{(1)}_{\mathbf{p}}(\alpha,\mathbf{n}). \tag{3.8}$$

The operators in (3.5) must be transformed according to the rule $D^{\dagger}(\mathbf{n})\,\mathbf{J}\,D(\mathbf{n}) = \widetilde{\mathbf{J}}$.

Applying (2.4), (2.5), we can express the functions $\xi^{(1)}_{\mathbf{p}}(\alpha,\mathbf{n})$ by means of the operators

$$\begin{aligned} A(\alpha,\mathbf{n}): \ = & \left[1 - \frac{i}{\alpha}\tau - \frac{1+i\alpha+\tau}{\alpha(\alpha-i)}2\mathbf{s}\cdot\mathbf{L} + \frac{i\tau+\alpha}{\alpha^2(\alpha-i)}\mathbf{L}^2 - \frac{2}{\alpha(\alpha-i)}(\mathbf{s}\cdot\mathbf{L})^2\right]\exp(i\frac{\partial}{\partial\alpha}) \\ & + \left[1 + \frac{i}{\alpha}\tau\right]\exp(-i\frac{\partial}{\partial\alpha}) + 2 - \frac{2i}{\alpha-i}\mathbf{s}\cdot\mathbf{L}, \end{aligned} \tag{3.9}$$

$$\xi^{(1)}_{\mathbf{p}}(\alpha,\mathbf{n}) = D^{\dagger}(\mathbf{n})\,A(\alpha,\mathbf{n})\,\frac{m}{2}\frac{\xi^{(0)}_{\mathbf{p}}(\alpha,\mathbf{n})}{p_0+m}, \tag{3.10}$$

where $\tau := 1 - (\mathbf{s}\cdot\mathbf{n})^2$. Using the equation

$$H^{(1)}(\alpha,\mathbf{n})\, D^{\dagger}(\mathbf{n})\, A(\alpha,\mathbf{n}) = D^{\dagger}(\mathbf{n})\, A(\alpha,\mathbf{n})\, H^{(0)}(\alpha,\mathbf{n}), \tag{3.11}$$

we have

$$\begin{aligned} H^{(1)}(\alpha,\mathbf{n}) = m \Big[& \cosh(i\frac{\partial}{\partial\alpha}) + \frac{i\alpha+\widetilde{\tau}}{\alpha(\alpha-i)}\sinh(i\frac{\partial}{\partial\alpha}) + \frac{\widetilde{\tau}}{\alpha^2}\exp(-i\frac{\partial}{\partial\alpha}) + \\ & \frac{(\alpha^2+\widetilde{\tau})\widetilde{\mathbf{J}}^2}{2\alpha^2(\alpha^2+1)}\exp(i\frac{\partial}{\partial\alpha}) - \frac{(\widetilde{\mathbf{s}}\cdot\widetilde{\mathbf{L}}+2)\widetilde{\tau}}{\alpha^2+1} - \frac{\widetilde{\tau}(\widetilde{\mathbf{s}}\cdot\widetilde{\mathbf{L}}+2)}{\alpha^2}\Big]. \end{aligned} \tag{3.12}$$

In the nonrelativistic limit, with the notation $q = \alpha/m$, we have

$$H^{(1)}(q,\mathbf{n}) - m \quad \rightarrow \quad -\frac{1}{2mq^2}\frac{\partial}{\partial q}q^2\frac{\partial}{\partial q} + \frac{\widetilde{\mathbf{J^2}} + 2[\widetilde{\tau} - (\widetilde{\mathbf{s}}\cdot\widetilde{\mathbf{J}})\widetilde{\tau} - \widetilde{\tau}(\widetilde{\mathbf{s}}\cdot\widetilde{\mathbf{J}})]}{2mq^2}. \tag{3.13}$$

One can determine the momentum operator either by means of the commutation relations of the Poincaré algebra $[\widetilde{\mathbf{N}}, H^{(1)}(\alpha,\mathbf{n})] = -i\mathbf{P}^{(1)}(\alpha,\mathbf{n})$, or by the equations

$$\mathbf{P}^{(1)}(\alpha,\mathbf{n})\, D^{\dagger}(\mathbf{n})\, A(\alpha,\mathbf{n}) = D^{\dagger}(\mathbf{n})\, A(\alpha,\mathbf{n})\, \mathbf{P}^{(0)}(\alpha,\mathbf{n}), \tag{3.14}$$

$$\begin{aligned} P_3^{(1)}(\alpha,\mathbf{n}) = n_3 H^{(1)}(\alpha,\mathbf{n}) - m \Big[& (\frac{\alpha(1-\widetilde{\tau})}{(\alpha^2+1)} + \frac{\widetilde{\tau}}{\alpha})\exp(i\frac{\partial}{\partial\alpha})N_3 + \\ & \frac{1-\widetilde{\tau}+s_3L_3}{\alpha^2+1}\exp(i\frac{\partial}{\partial\alpha}) + \frac{(\widetilde{\mathbf{s}}\times\mathbf{n})_3}{\alpha}(1-\frac{\widetilde{\tau}}{\alpha+i})\Big]. \end{aligned} \tag{3.15}$$

The operators $H^{(1)}(\alpha,\mathbf{n})$, $\mathbf{P}^{(1}(\alpha,\mathbf{n})$ can be identified as operators of the massive relativistic spin 1 particle in the Schrödinger picture in $(\alpha,\mathbf{n})$-representation. The state functions in the Schrödinger picture can be found by means of the Fourier expansion in the Heisenberg picture (3.2) and the transformation (1.1):

$$\Psi_{\mu}^{(1)}(\alpha,\mathbf{n},t,\mathbf{x}) = \frac{1}{(2\pi)^{3/2}}\sum_{\mu'=-1}^{1}\int\frac{d\mathbf{p}}{p_0}\,\xi^{(1)}_{\mathbf{p}\mu\mu'}(\alpha,\mathbf{n})\,\exp[-i(p\cdot x)]\Psi^{(1)}_{\mu'}(\mathbf{p}). \tag{3.16}$$

In this case the Schrödinger equation is valid

$$i\frac{\partial}{\partial t}\Psi^{(1)}(\alpha,\mathbf{n},t,\mathbf{x}) = H^{(1)}(\alpha,\mathbf{n})\Psi^{(1)}(\alpha,\mathbf{n},t,\mathbf{x}), \tag{3.17}$$

as well the equation in the spatial derivatives

$$-i\nabla_{\mathbf{x}}\Psi^{(1)}(\alpha,\mathbf{n},t,\mathbf{x}) = \mathbf{P}^{(1)}(\alpha,\mathbf{n})\Psi^{(1)}(\alpha,\mathbf{n},t,\mathbf{x}). \tag{3.18}$$

IV. PARTIAL-WAVE EQUATIONS

To determine the partial-wave equations in the $(\alpha, \mathbf{n})$-representation in Heisenberg picture, we first use the spherical spinors $\Omega_{\jmath\ell m}(\mathbf{n}_p)$, $\Omega_{\jmath\ell m}(\mathbf{n})$ being the eigenfunctions of the operators $\mathbf{s}\cdot\mathbf{L}(\mathbf{p})$ and $\mathbf{s}\cdot\mathbf{L}$. They have the same form as in the nonrelativistic formalism. The equation (3.8) permits factorization by introducing the spinors $\widetilde{\Omega}_{\lambda\jmath\ell m}(\mathbf{n}) := D^{\dagger}(\mathbf{n})\cdot\Omega_{\jmath\ell m}(\mathbf{n})$. If we introduce $D^{\dagger}_{11}(\mathbf{n}) = (1+n_3)/2$, $D^{\dagger}_{10}(\mathbf{n}) = -n_-/\sqrt{2}$, $D^{\dagger}_{1-1}(\mathbf{n}) = n_-^2/2(1+n_3)$, $(s_3)_{11} = 1$, then

$$\widetilde{J}^{(1)}_3 = L_3 + s_3, \quad \widetilde{J}^{(1)}_- = L_- + s_3 n_-/(1+n_3), \quad \widetilde{J}^{(1)}_+ = L_+ + s_3 n_+/(1+n_3), \tag{4.1}$$

with $\widetilde{\mathbf{s}\cdot\mathbf{L}}\,\widetilde{\Omega}_{\lambda\jmath\ell m}(\mathbf{n})=\beta\,\widetilde{\Omega}_{\lambda\jmath\ell m}(\mathbf{n})$, $\beta=\jmath(\jmath)+1-\ell(\ell-1)-2$.

Let us integrate the expression

$$\xi^{(1)}_{\mathbf{p}}(\alpha,\mathbf{n})\Omega_{\jmath\ell m}(\mathbf{n}_p) = D^{\dagger(1)}(\mathbf{n})\,A(\alpha,\mathbf{n})\,\frac{m}{2}\frac{\xi^{(0)}_{\mathbf{p}}(\alpha,\mathbf{n})}{p_0+m}\Omega_{\jmath\ell m}(\mathbf{n}_p), \tag{4.2}$$

over the angular variables of the $\mathbf{n}_p$ vectors. The matrix elements obtained in this way can be written in the form

$$\widetilde{A(\alpha,\mathbf{n})}\,\frac{m}{2}\frac{\mathcal{P}^{(0)}_l(\cosh\chi,\alpha)}{p_0+m}\cdot 4\pi i^l\widetilde{\Omega}_{\lambda\jmath\ell m}(\mathbf{n}. \tag{4.3}$$

We define the partial functions for the particles with spin 1 $\mathcal{P}^{(1)}_{\lambda\jmath\ell}(\cosh\chi,\alpha)$ as coefficients that stand in front of $4\pi i^l\widetilde{\Omega}_{\lambda\jmath\ell m}(\mathbf{n})$ in the expression (4.3). These can be expressed in terms of the functions $\mathcal{P}^{(0)}_l(\cosh\chi,\alpha)$ (see Appendix), $b(\chi) := 1/2(\cosh\chi+1)$:

1) for $\jmath = \ell+1$,

$$\mathcal{P}^{(1)}_{\lambda\jmath\ell}(\cosh\chi,\alpha) = b(\chi)\left[\frac{(\alpha-il)(\alpha-il-i)}{\alpha(\alpha-i|\lambda|)}\exp(i\frac{\partial}{\partial\alpha}) + \frac{2(\alpha-il-i)}{\alpha-i} + \frac{\alpha+i-|\lambda|}{\alpha}\exp(-i\frac{\partial}{\partial\alpha})\right]\mathcal{P}^{(0)}_l(\cosh\chi,\alpha); \tag{4.4}$$

2) for $\jmath = \ell-1$,

$$\mathcal{P}^{(1)}_{\lambda\jmath\ell}(\cosh\chi,\alpha) = b(\chi)\left[\frac{(\alpha+il)(\alpha+il+i)}{\alpha(\alpha-i|\lambda|)}\exp(i\frac{\partial}{\partial\alpha}) + \frac{2(\alpha+il)}{\alpha-i} + \frac{\alpha+i-|\lambda|}{\alpha}\exp(-i\frac{\partial}{\partial\alpha})\right]\mathcal{P}^{(0)}_l(\cosh\chi,\alpha); \tag{4.5}$$

3) for $\jmath = \ell$ and $|\lambda| = 1$,

$$\mathcal{P}^{(1)}_{\lambda\jmath\ell}(\cosh\chi,\alpha) = b(\chi)\left[\frac{\alpha(\alpha+i)-\jmath(\jmath+1)}{\alpha(\alpha-i)}\exp(i\frac{\partial}{\partial\alpha}) + \frac{2\alpha}{\alpha-i} + \exp(-i\frac{\partial}{\partial\alpha})\right]\mathcal{P}^{(0)}_{l}(\cosh\chi,\alpha); \tag{4.6}$$

4) for $\jmath = \ell$ and $\lambda = 0$, $\tilde{\Omega}_{\lambda\jmath\ell m}(\mathbf{n})$=0, $\mathcal{P}^{(1)}_{\lambda\jmath\ell}(\cosh\chi,\alpha)$=0.

For the functions $\mathcal{P}^{(1)}_{\lambda\jmath\ell}(\cosh\chi,\alpha)$ we have

$$H^{(1)}(\alpha,\jmath,\lambda)\,\mathcal{P}^{(1)}_{\lambda\jmath\ell}(\cosh\chi,\alpha) = p_0\,\mathcal{P}^{(1)}_{\lambda\jmath\ell}(\cosh\chi,\alpha), \tag{4.7}$$

where

$$H^{(1)}(\alpha,\jmath,\lambda):\ = m\left[\frac{\alpha}{2(\alpha-i)} + \frac{\tilde{\tau}}{2(\alpha-i)} + \frac{\jmath(\jmath+1)}{2(\alpha^2+1)}(1+\frac{\tilde{\tau}}{\alpha^2})\right]\exp(i\frac{\partial}{\partial\alpha}) + m\left[\frac{\alpha-2i}{2(\alpha-i)}(1+\frac{\tilde{\tau}}{\alpha^2})\right]\exp(-i\frac{\partial}{\partial\alpha}) + m\begin{pmatrix} 0 & -\frac{\beta+1}{\alpha^2+1} & 0 \\ -\frac{\beta+2}{2\alpha^2} & 0 & -\frac{\beta+2}{2\alpha^2} \\ 0 & -\frac{\beta+1}{\alpha^2+1} & 0 \end{pmatrix}. \tag{4.8}$$

V. CONCLUSION

The four-dimensional generalization of the Heisenberg/Schrödinger picture introduces new features into the nature of the description of particle states. In the framework of this approach the plane wave $\sim$ const$\cdot\exp[-i(p\cdot x)]$ in their original sense as the stationary states of a particle cannot appear in the mathematical formalism of the quantum theory. The consistent determination of the wave functions of a particle in the Schrödinger picture requires the use of the wave functions in the momentum representation or of the matrix elements of the fundamental series of unitary representations of the Lorentz group.

The found Hamiltonian for spin 1 - particle is a differential-difference operator. The system of eigenfunctions is expressed through the eigenfunction of the particle with spin zero.

We hope that the formalism that has been developed here will be employed for solving problems in relativistic quantum physics.

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APPENDIX: ORTHOGONALITY AND COMPLETENESS

The partial expansion for the function $\xi^{(0)}(\mathbf{p},\alpha,\mathbf{n})$ has the form
$(p_0/m := \cosh\chi, \quad \mathbf{n}_p := \mathbf{p}/|\mathbf{p}\ |, \quad \mathbf{n}_p := (\sin\theta_p\cos\varphi_p, \sin\theta_p\sin\varphi_p, \cos\theta_p), \quad |\mathbf{p}|/m = \sinh\chi)$,

$$\xi^{(0)}(\mathbf{p},\alpha,\mathbf{n}) = \sum_{l=0}^{\infty}(2l+1)i^l\mathcal{P}_l^{(0)}(\cosh\chi,\alpha)P_l(\mathbf{n_p}\cdot\mathbf{n}), \tag{A1}$$

with the functions

$$\mathcal{P}_l^{(0)}(\cosh\chi,\alpha) = (-i)^l\sqrt{\frac{\pi}{2\sinh\chi}}\,\frac{\Gamma(i\alpha+l+1)}{\Gamma(i\alpha+1)}\mathcal{P}^{-1/2-l}_{-1/2+i\alpha}(\cosh\chi), \tag{A2}$$

$$\mathcal{P}_l^{(0)}(\cosh\chi,\alpha) = i^l\frac{\Gamma(i\alpha+1)}{\Gamma(-i\alpha+l+1)}(\sinh\chi)^l(\frac{d}{d\sinh\chi})^l\mathcal{P}^{(0)}_{(0)}(\cosh\chi,\alpha), \tag{A3}$$

$$\mathcal{P}^{(0)}_{(0)}(\cosh\chi,\alpha) = \frac{\sin(\alpha\chi)}{\alpha\sinh\chi}. \tag{A4}$$

The orthogonality and completeness conditions for the functions $\xi^{(s)}(\mathbf{p},\alpha,\mathbf{n})$ have the form

$$\frac{1}{(2\pi)^3}\sum_{\nu=-s}^{s}\int(\nu^2+\alpha^2)d\alpha\,d\omega_{\mathbf{n}}\,\xi^{\dagger(s)}_{\mu\nu}(\mathbf{p},\alpha,\mathbf{n})\,\xi^{(s)}_{\nu\sigma}(\mathbf{p_1},\alpha,\mathbf{n}) = \delta_{\mu\sigma}\delta^{(3)}(\mathbf{p}-\mathbf{p_1})\sqrt{1+\mathbf{p}^2/m^2}, \tag{A5}$$

$$\frac{1}{(2\pi)^3}\sum_{\nu=-s}^{s}\int\frac{d\mathbf{p}}{p_0}\,\xi^{(s)}_{\mu\nu}(\mathbf{p},\alpha,\mathbf{n})\,\xi^{\dagger(s)}_{\nu\sigma}(\mathbf{p_1},\alpha_1,\mathbf{n_1}) = \delta_{\mu\sigma}\frac{\alpha^2}{\mu^2+\alpha^2}\,\delta^{(3)}(\mathbf{n}-\mathbf{n_1})\,\delta(\alpha-\alpha_1). \tag{A6}$$

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